Fast Algorithms Of Multidimensional Discrete Nonseparable $K$–Wave Transforms

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Abstract

Fast algorithms for a wide class of non–separable $n$–dimensional (nD) discrete unitary $K$–transforms (DKT) are introduced. They need less 1D DKTs than in the case of the classical radix–2 FFT–type approach. The method utilizes a decomposition of the nD $K$–transform into the product of a new nD discrete Radon transform and of a set of parallel/independ 1D $K$–transforms. If the nD $K$–transform has a separable kernel (e.g., the case of the discrete Fourier transform) our approach leads to decrease of multiplicative complexity by the factor of $n$ comparing to the classical row/column separable approach.

Keywords

Multidimensional Radon, Fourier, Hartley transforms, Nussbaumer transform, fast algorithms

I. Introduction

Discrete unitary transforms such as, for example, the discrete Fourier $F_{N^n} := \left[ \operatorname{cis} \left( \frac{2\pi}{N} (k|i) \right) \right]$, Hartley $H_{N^n} := \left[ \operatorname{cas} \left( \frac{2\pi}{N} (k|i) \right) \right]$ transforms, the discrete cosine–I $C_{N^n}^I := \left[ \cos \left( \frac{N-1}{N} (k|i) \right) \right]$ and sine $S := \left[ \sin \left( \frac{\pi}{N} (k|i) \right) \right]$ transforms, etc., form a widely used tool in digital signal processing, where $\operatorname{cis}(x) := \cos(x) + jsin(x) = \exp(jx)$, $\operatorname{cas}(x) := \cos(x) + \sin(x)$, $(k|i) := k_1 i_1 + \ldots + k_n i_n$ is the inner product in the nD cubes $Z_N^n$. Here, $Z_N$ is the ring of integers modulo $N$. Many algorithms and VLSI architectures for discrete unitary transforms have been proposed aiming to improve the computation speed and to reduce the hardware complexity [1]–[3], [4], [5]–[18]. They can be classified into the following four categories: 1) matrix factorization, 2) recursive computation, 3) indirect computation, 4) systolic structure implementation.

In this paper we derive a new matrix factorization scheme (fast parallel/independ algorithm) for an arbitrary nD discrete unitary transform (DUT) with the kernel of the following type: $K_{N^n}(k|i) := K_N(k_1 i_1 + \ldots + k_n i_n)$, where $K_N(\cdot)$ is the kernel of the 1D DUT. Such transforms in this paper are called $K$–transforms. The $K$–kernel does not in general possess the separability property, i.e., in general case $K_{N^n}(k_1 i_1 + \ldots + k_n i_n) \neq K_N(k_1 i_1) \ldots K_N(k_n i_n)$ and, as a result, nD $K$–transform is not the tensor product of 1D $K$–transforms, i.e., $K_{N^n} \neq K_N \otimes \ldots \otimes K_N$. Therefore, in this case, nD $K$–transform does not possess the radix–$N$ algorithm. Our method utilizes a decomposition of an nD $K$–transform into the product of a new nD discrete Radon transform and of a set independ/parallel 1D $K$–transforms. If the nD $K$–transform has a separable kernel (for example, the discrete Fourier transform), then our approach leads the algorithm with a lower multiplication complexity by the factor $n$ comparing to the classical row/column Cooley’s–Tukey’s approach [1].

The paper is organized as follows. In section 2 definitions of the $K$–transform and of the classical Radon transform are given. In section 3 we introduce new nD $N^n$–point discrete Radon transforms (DRTs). Fast algorithms of such DRTs are considered in the sections IV and V. In the section IV we consider the simplest case: 2D $q^2$–point discrete $K$– and Radon transforms, where $q$ is a prime. Fast algorithms of such DRTs are based on using of Winograd’s fast Fourier Transform (WFFT)–structure for its realization (see [1]–[3]). They have minimal numbers of
multiplications among different other algorithms. Unfortunately, this minimality does not yet guarantee an efficient implementation. Moreover, most of problems arising in "real life" require efficient fast algorithms not only for prime but for composite (special case, powers of 2) sizes. Fast algorithm for \((q^n)^n\)-points nD DRT, is considered in section V. In this section fast algorithms for the direct and inverse DRT and for the discrete \(K\)-transform for arbitrary \(N\) are presented too.

II. Multidimensional \(K\)- and Radon Transforms

Let \(\mathbb{R}^n\) be an nD space (nD signal space) consisting of column vectors \(x := (x_1, \ldots, x_n)^t\) with components \(x_1, \ldots, x_n \in \mathbb{R}\), where "\(t\)" is the symbol of transposition. Let \(\mathbb{R}^{*n}\) be a dual space (nD spectral space) consisting of row vectors \(\omega := (\omega_1, \ldots, \omega_n)\).

\[F(\omega) = \mathcal{K}_{\mathbb{R}^n}\{f(x)\} := \int_{\mathbb{R}^n} f(x) K(|\omega|x) \, dx, \quad (1)\]

\[f(x) = \mathcal{K}_{\mathbb{R}^n}^{-1}\{F(\omega)\} := \int_{\mathbb{R}^n} F(\omega) K(|\omega|x) \, d\omega, \quad (2)\]

are called the direct and inverse nD \(K\)-transforms, where \(|\omega|x := \sum_{i=1}^{n} \omega_i x_i\), \(dx := dx_1 \ldots dx_n\), \(d\omega := d\omega_1 \ldots d\omega_n\).

The function \(K(|\omega|x)\) is the plane \(K\)-wave, which is propagated along the vector \(\omega\). In all points of the hyperplane \(|\omega|x = p\) this \(K\)-wave has a constant "phase" equals \(p\). Denote by \(\mathcal{P}^n\) the space of all hyperplanes \(\pi_n(\alpha, p) : \langle \alpha|x\rangle = p\), where \(p \in \mathbb{R}^+, \alpha \in \Sigma^*_n-1\) and \(\Sigma^*_n-1\) is the \((n-1)\)-dimensional unit sphere in \(\mathbb{R}^n\).

Definition 1: The unitary operators \(\mathcal{K}_{\mathbb{R}^n}\) and \(\mathcal{K}_{\mathbb{R}^n}^{-1}\), acting by the following rules:

\[\hat{\mathcal{K}}_{\mathbb{R}^n}\{f(x)\} := \hat{f}(\alpha, p) = \int_{\langle \alpha|x\rangle = p} f(x) \, dx = \int_{\mathbb{R}^n} f(x) \delta(p - \langle \alpha|x\rangle) \, dx, \quad (3)\]

e.i. \(\hat{f}(\alpha, p)\) is equal to the integral of the function \(f(x)\) along the hyperplane \(\pi_n(\alpha, p)\).

The Radon transform is closely related to the multidimensional \(K\)-transform. Indeed, if \(\omega = |\omega|\alpha = a\alpha\), where \(a = |\omega|\) and \(\alpha \in \Sigma^*_n-1 \subset \mathbb{R}^n\), then, in order to calculate \(F(\omega) = F(a\alpha)\),

\[F(a\alpha) = \mathcal{K}_{\mathbb{R}^n}\{f(x)\} = \int_{\mathbb{R}^n} f(x) K(a\langle \alpha|x\rangle) \, dx = \int_{-\infty}^{+\infty} \left( \int_{\langle \alpha|x\rangle = p} f(x) \, dx \right) K(ap) \, dp = \int_{-\infty}^{+\infty} \hat{f}(\alpha, p) K(ap) \, dp := \mathcal{K}_R^\alpha \{\hat{f}(\alpha, p)\}. \quad (4)\]

This connection is called a generalized central slice theorem for a continuous nD \(K\)-transform.

Theorem 3: [20]–[21] The 1D \(K\)-transform \(\mathcal{K}_R^\alpha\) of a projection \(\hat{f}(\alpha, p)\) along the ray \(\text{Ray}_R^\alpha(\alpha)\) is the central slice of the nD \(K\)-spectrum \(F(a\alpha)\).

It means that the nD transform \(\mathcal{K}_R^n\) is a composition of the Radon transform \(\mathcal{R}_{\mathbb{R}^n}\) and of a family of 1D \(K\)-transforms \(\mathcal{K}_R^\alpha = \bigoplus_{\alpha \in \Sigma^*_n-1} \mathcal{K}_R^\alpha\mathcal{R}_{\mathbb{R}^n}\). The cardinality of 1D \(K\)-transforms from this family equals to the cardinality of the sphere \(\Sigma^*_n-1\). It is convenient to introduce the following terminology. We will call the unit sphere \(\Sigma^*_n-1\) the star and denote it by \(\text{St} := \Sigma^*_n-1\).

If \(\alpha \in \text{St}\) then the set \(\text{Ray}_R^\alpha(\alpha) := \{a\alpha|a \in \mathbb{R}\}\) is called the \(\mathbb{R}\)-ray associated with a point \(\alpha\).
The remarkable property of the star $\text{St}$ is that all the rays $\text{Ray}_R(\alpha)$, $\alpha \in \text{St}$ cover the whole spectrum domain $\mathbb{R}^n$, i.e., $\mathbb{R}^n = \bigcup_{\alpha \in \text{St}} \text{Ray}(\alpha)$.

The Radon transform (RT) and its ill–conditioned inverse were first formulated by J. Radon in 1917 [19]. Currently, the RT is widely used in a variety of applications including tomography, ultrasound, optics, and geophysics, to name a few.

Discrete versions of the classical RT are being used in signal processing and there is an extensive literature devoted to this subject. Procedures which are discrete versions of the RT are known since 1917 [19]. Currently, the RT is widely used in a variety of applications including tomography, ultrasound, optics, and geophysics, to name a few.

In the next section, we introduce a new direct and inverse DRTs and show that they admit fast computation by the fast Nussbaumer polynomial transform (NPT) [4]. Using the discrete versions we derive a new fast algorithm for the multidimensional $K$–Transform.

III. DISCRETE $K$– AND RADON TRANSFORMS

The purpose of this section is to describe a new discrete Radon transform (DRT). Let $\text{Sig}(\mathbb{Z}_N^N) := \mathbb{Z}_N^N$ be an nD discrete cube (nD signal domain). Its elements are column–vectors $i := (i_1, \ldots, i_n)^t$, $i_1, \ldots, i_n \in \mathbb{Z}_N$. Let $\text{Sp}(\mathbb{Z}_N^N) := \mathbb{Z}_N^{*n}$ be the dual cube (nD spectral domain) consisting of row vectors $k := (k_1, k_2, \ldots, k_n)$, with components $k_1, \ldots, k_n \in \mathbb{Z}_N^*$. 

**Definition 4:** The unitary operators $K_N^n$ and $K_N^{-1}$, acting by the following rules:

$$F(k) = K_N^n \{ f(i) \} = \sum_{i \in \mathbb{Z}_N^n} f(i)K(\langle k | i \rangle),$$

(5)

$$f(i) = K_N^{-1} \{ F(k) \} = \sum_{k \in \mathbb{Z}_N^{*n}} F(k)K(\langle k | i \rangle),$$

(6)

are called the direct and inverse discrete nD $K$– transforms (DKT).

**Definition 5:** [38]–[41] A minimal set $\text{St} = \{ \alpha \}$ of vectors $\alpha \in \text{Sp}(\mathbb{Z}_N)$ is called the nD star if all the rays $\text{Ray}_{\mathbb{Z}_N^N}(\alpha) := \{ a \cdot \alpha \mid a \in \mathbb{Z}_N^1 \}$ cover the spectral domain $\text{Sp}(\mathbb{Z}_N)$, i.e., $\bigcup_{\alpha \in \text{St}} \text{Ray}_{\mathbb{Z}_N^N}(\alpha) = \text{Sp}(\mathbb{Z}_N)$. If $\text{St} = \bigcup_r \text{St}_r$ and $\text{St}_r \cap \text{St}_s = \emptyset$, $\forall r, s$, then every subset $\text{St}_r$ is called the substar of the star $\text{St}$.

The rays of this star run through all the points of spectral domain. By this reason we can write the following expression for spectral values lying on all the rays $\text{Ray}(\alpha)$:

$$F(\text{Ray}(\alpha)) = F(a\alpha) = \sum_{\mathbb{Z}_N^N} f(i)K(a/\alpha | i) = \sum_{p=0}^{N-1} \left( \sum_{|i|=p} f(i) \right) K(ap),$$

(7)

or

$$F(a\alpha) = \sum_{p=0}^{N-1} \hat{f}(\alpha, p)K(ap), \quad \text{where} \quad \hat{f}(\alpha, p) = R_N^n \{ f(i) \} = \sum_{|i|=p} f(i).$$

(8)
Definition 6: [38]–[45] The function 

\[ \hat{f}(\alpha, p) = R_{N^n}\{f(i)\} = \sum_{\langle \alpha | i \rangle = p} f(i), \]

which is equal to the sum of values of the signal \( f(i) \) on all discrete hyperplanes \( \langle \alpha | i \rangle = p \) (\( \alpha \in \text{ST} \)) is called the discrete Radon transform (DRT) of \( f(i) \).

There exists another definitions of the direct discrete Radon transform (see [32]–[37]). Our definition gives an invertible discrete Radon transform; it connected with Nussbaumer polynomial transform [4] and by this reason has fast algorithm.

The Expressions (8) mean that \( nD K \)-transform \( K_{N^n} \) is a composition of \( nD \) DRT \( R_{N^n} \) and a family of parallel/independ 1D \( K \)-transforms

\[ K_{N^n} = \left[ \bigoplus_{\alpha \in \text{ST}} K_N^\alpha \right] R_{N^n} = \begin{bmatrix} K_N^\alpha_1 & & \\ & K_N^\alpha_2 & \\ & & \vdots \\ & & K_N^\alpha_{\text{St}} \end{bmatrix} R_{N^n}, \tag{9} \]

where \( \left[ \bigoplus_{\alpha \in \text{ST}} K_N^\alpha \right] \) is a block–diagonal \((N^n \times |\text{ST}|N)\)-transform and \( R_{N^n} \) is the discrete Radon \((|\text{ST}|N \times N^n)\)-transform. The total number of 1D \( K \)-transforms \( K_N^\alpha \) is equal to the cardinality of the star \( |\text{ST}| \). Each 1D \( K \)-transform \( K_N^\alpha \) acts along the star ray \( \text{Ray}(\alpha) \). This connection will be called as a generalized central slice theorem for discrete \( K \)-transforms [45].

Theorem 7: The 1-D \( K \)-transform \( K_N^\alpha \) of the projection \( \hat{f}(\alpha, p) \) along the ray \( \text{Ray}_{Z_N^2}(\alpha) \) is the central slice of \( nD K \)-spectrum, i.e.,

\[ F(u\alpha) = \left[ \bigoplus_{\alpha \in \text{ST}} K_N^\alpha \right] \hat{f}(\alpha, p). \]

In further we shall need in full and reduced matrix transform. A square \( N \times N \) matrix transform \( K_{N \times N} \) is called the full transform and it is called the reduced transform if it has not several lines, i.e. it is \( M \times N \) matrix transform \( K_{M \times N} \), where \( M < N \).

Note that if \( nD K \)-transform \( K_{N^n} \) is separable on \( Z_N^2 \), then the classical “row/column” algorithm reduces this transform to \( nN^{n-1} \) 1D \( K \)-transforms \( K_N \) of the length \( N \) [1]–[3]. Our main results are presented in the following three theorems 11–13 which summarize and extend our results published in [38]–[45] and [46]–[49]. These Theorems show that \( nD K \)-transform \( K_{N^n} \) is a composition of \( nD \) DRT \( R_{N^n} \) and a set of \(|\text{ST}| \approx N^{n-1}\) parallel/independ 1D \( K \)-transforms \( K_{N^n} = \left[ \bigoplus_{\alpha \in \text{ST}} K_N^\alpha \right] R_{N^n} \), leading to decrease of a total number of 1D \( K \)-transforms by the factor of \( n \) comparing to the classical “row/column” approach.

IV. FAST 2D RADON AND \( K \)-TRANSFORMS

In this section we present the fast discrete Radon and \( K \)-transforms. The schemes of the fast algorithms are based on the Nussbaumer’s polynomial transform [4]. The form of the star \( \text{ST} \) greatly depends on the canonical decomposition of \( N_1, N_2, \ldots, N_n \) into prime factors. First, for simplicity, we consider the case of the 2D transforms, for \( N_i = q, i = 1, 2, \ldots, n \), where \( q \) is a prime.

The 2D DKT on the ring \( Z_q^2 \) is given by

\[ F(k_1, k_2) = \sum_{n_1=0}^{q-1} \sum_{n_2=0}^{q-1} f(n_1,n_2)K(n_1k_1 + n_2k_2), \quad k_1, k_2 \in Z_q. \]
Theorem 8: If \( N = q \) is a prime, then the star consists of two substars \( \text{St} = \text{St}_1 \cup \text{St}_2 = \{ \alpha_1 \} \cup \{ \alpha_2 \} \), where \( \alpha_1 := (k_1, 1) \), \( \alpha_2 = (1, 0) \). In this case the 2D \( K \)-transform \( K_{(q)^2} \) can be computed by \((q^2-1)/(q-1) = q+1\) 1D \( K \)-transforms \( K_q \) and by one discrete 2D Radon transform:

\[
K_{(q)^2}\{f(i_1, i_2)\} = \left[ K_q^{\alpha_2} \oplus \bigoplus_{\alpha_1 \in \text{St}_1} \tilde{K}_q^{\alpha_1} \right] \mathcal{R}_{(q)^2}\{f(i_1, i_2)\},
\]

where \( K_q \) and \( \tilde{K}_q \) are full and 0-reduced (without the first line) 1D \( K_q \)-transforms, respectively.

Proof: The Galois field \( Z_q \) is the union of the multiplicative group \( MZ_q \) and the zero \( \{0\} : Z_q = MZ_q \cup \{0\} \). We can present the spectral domain as the following union:

\[
\text{Sp}(Z_q) = Z_q \times Z_q = Z_q \times (MZ_q \cup \{0\}) = (Z_q \times MZ_q) \cup (Z_q \times \{0\}) =
\]

\[
= \text{Ray}_{MZ_q}(\{\alpha_1\}) \cup \text{Ray}_{Z_q}(\{\alpha_2\}) = \text{Ray}_{MZ_q}(\text{St}_1) \cup \text{Ray}_{Z_q}(\text{St}_2). \tag{11}
\]

Note that all the rays \( \text{Ray}_{Z_q}(\{\alpha_1\}) \) and \( \text{Ray}_{Z_q}(\{\alpha_2\}) \) over \( Z_q \) are intersected in the origin. Hence, we have to take in (11) the ray \( \text{Ray}_{Z_q}(\text{St}_2) \) over \( Z_q \) and take all the rays \( \text{Ray}_{Z_q}(\text{St}_1) \) only over \( MZ_q \). Using (9) and (11), we obtain (10). This proof is based on the results obtained in [38]–[40], where the expression (10) was first time presented for 2D discrete Fourier and Radon transforms on a 2D rectangle lattice. The generalizations of these results for an arbitrary 2D lattices (so-called Fourier–Mersereau and Radon–Mersereau transforms) can be found in [42] and [43]. Analogous results (for 2D rectangle lattice) are presented in [5]–[18]. In [7],[10],[14]–[15] these results are used to develop the fast 2D Hartley and cosine transforms.

Example 9: For the visual representation, we can imagine the star \( \text{St} \) in the form of a discrete circle on which all the points \( \alpha \in \text{St} \) lie equidistantly. The origin is located in the centre of this circle. Then, the spectral domain \( \text{Sp}(Z_N) \) can be represented as a star with rays passing through all the points \( \alpha \in \text{St} \). Let \( q = 3 \), then the star \( \text{St} \) consists of 4 points \( \text{St} = \text{St}_1 \cup \text{St}_2 = \{(2, 1), (1, 1), (0, 1)\} \cup \{(1, 0)\} \) and the spectral domain \( \text{Sp}(Z_3) \) is covered by 4 rays:

\[
\begin{align*}
\text{St} &= (1, 2) \quad (0, 0) \quad (0, 1) \quad (1, 1) \\
\text{Sp}(Z_3) &= (1, 2) \quad (2, 1) \quad (0, 0) \quad (0, 1) \quad (0, 2) \quad (1, 1) \quad (2, 2)
\end{align*}
\]

From the Theorem 8 one can see that DRT split in two discrete subtransforms (sub DRT):

\[
\tilde{f}(\alpha_1, p) = \sum_{k_1i_1 + i_2 = p} f(i_1, i_2), \quad \tilde{f}(\alpha_2, p) = \sum_{i_2 = 0}^{q-1} f(p, i_2).
\]

The question is: how to calculate 2D DRT \( \tilde{f}(\alpha_1, p) \) quickly? To do this we interpret the 2D scalar-valued signal

\[
f(i_1, i_2) = \begin{bmatrix}
f(0, 0) & f(0, 1) & \ldots & f(0, q - 1) \\
f(1, 0) & f(1, 1) & \ldots & f(1, q - 1) \\
\vdots & \vdots & \ddots & \vdots \\
f(q - 1, 0) & f(q - 1, 1) & \ldots & f(q - 1, q - 1)
\end{bmatrix}
\]
as the following 1D polynomial–valued signal:

\[
\hat{f}_z(i_1) = \sum_{i_2=0}^{q-1} f(i_1, i_2) z^{i_2} = \begin{bmatrix}
  f(0,0) z^0 \\
  f(0,1) z^1 \\
  \vdots \\
  f(0, q-1) z^{q-1}
\end{bmatrix}, \begin{bmatrix}
  f(1,0) z^0 \\
  f(1,1) z^1 \\
  \vdots \\
  f(1, q-1) z^{q-1}
\end{bmatrix}, \ldots, \begin{bmatrix}
  f(q-1,0) z^0 \\
  f(q-1,1) z^1 \\
  \vdots \\
  f(q-1, q-1) z^{q-1}
\end{bmatrix} = \begin{bmatrix}
  f_z(0) \\
  f_z(1) \\
  \vdots \\
  f_z(q-1)
\end{bmatrix}.
\]

The components of the signal \( f_z(i_1) : \mathbb{Z}_q \longrightarrow \mathbb{R}[z]/(z^q - 1) \) have values from the polynomial ring \( \mathbb{R}[z]/(z^q - 1) \). The \( q \)D space of such signals will be denoted by \( L(\mathbb{Z}_q, \mathbb{R}[z]/(z^q - 1)) \). In this space we select \( q \) functions

\[
\mathcal{E}_z^{k_1}(i_1) = z^{k_1 i_1} : \mathbb{Z}_q \longrightarrow \mathbb{R}[z]/(z^q - 1), \quad k_1, i_1 = 0, \ldots, q - 1
\]

and introduce the scalar product \( (f_z(i_1)|g_z(i_1)) = \sum_{i_1=0}^{q-1} f_z(i_1) g_z^{-1}(i_1) \). The functions \( \mathcal{E}_z^{k_1}(i_1) \), \( k_1 = 0, \ldots, q - 1 \) are orthogonal with respect to this scalar product. Indeed,

\[
\langle \mathcal{E}_z^{k_1}(i_1)|\mathcal{E}_z^{l_1}(i_1) \rangle = \sum_{i_1=0}^{q-1} z^{k_1 i_1} z^{-l_1 i_1} = \sum_{i_1=0}^{q-1} z^{(k_1-l_1)i_1} = \begin{cases}
  q \mod(z^q - 1), & \text{if } k_1 = l_1, \\
  0 \mod(z^q - 1), & \text{if } k_1 \neq l_1.
\end{cases}
\]

Since the total number of these functions is equal to the dimension of the space \( L(\mathbb{Z}_q, \mathbb{R}[z]/(z^q - 1)) \), they form an orthogonal basis. Therefore, any signal of type (12) can be decomposed into a series

\[
f_z(i_1) = \mathcal{N}_q^{-1} \left\{ \hat{f}_z(k_1) \right\} = \sum_{k_1=0}^{q-1} \hat{f}_z(k_1) z^{-k_1 i_1}, \mod(z^q - 1), \quad (13)
\]

where

\[
\hat{f}_z(k_1) = \mathcal{N}_q \{ f_z(i_1) \} = \sum_{i_1=0}^{q-1} f_z(i_1) z^{k_1 i_1}, \mod(z^q - 1) \quad (14)
\]

is the polynomial–valued spectrum. The expressions (13) and (14) are called the inverse and direct Nussbaumer polynomial transforms [4].

Let us clarify the geometrical nature of the polynomial–valued spectrum. Substituting (12) into (14) we obtain

\[
\hat{f}_z(k_1) = \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} f(i_1, i_2) z^{k_1 i_1 + i_2} = \sum_{p=0}^{q-1} \left( \sum_{k_1 i_1 + i_2 = p} f(i_1, i_2) \right) z^p = \sum_{p=0}^{q-1} \hat{f}(\alpha_1, p) z^p = \left( \begin{array}{c}
  \hat{f}(\alpha_1, 0) z^0 \\
  \hat{f}(\alpha_1, 1) z^1 \\
  \vdots \\
  \hat{f}(\alpha_1, q-1) z^{q-1}
\end{array} \right) = \hat{f}_z(\alpha_1). \quad (15)
\]
where

\[ \hat{f}(\alpha_1, p) = \sum_{p=0}^{q-1} \hat{f}(\alpha_1, p)z^p = \sum_{k_1i_1 + i_2=p} f(i_1, i_2)z^p. \]

Thus, the coefficients of the polynomial \( \hat{f}_z(\alpha_1) \) are the spectral coefficients of the Radon transform \( \hat{f}(\alpha_2, p) \) of the initial signal \( f(i_1, i_2) \) on the hyperplanes \( \langle \alpha_1 | i \rangle = \{ k_1i_1 + i_2 = p \} \). In other words, in order to calculate DRT \( \hat{f}(\alpha_2, p) \) it is necessary to calculate the polynomial spectrum of the initial signal, which can be done using the fast Nussbaumer polynomial transform. Note that the NPT is an invertible transform (see (13)). Calculation of the inverse NPT is equivalent to calculation of the inverse DRT or it is equivalent to integration over the hyperplanes \( \langle \alpha_2^{-1} | i \rangle := \{-k_1i_1 + i_2 = p\} \).

**Example 10:** If \( q = 3 \), we have

\[
\begin{align*}
\hat{f}_z(0, 1) &= \begin{bmatrix} \hat{f}((0, 1), 0)z^0 \\ \hat{f}((0, 1), 1)z^1 \\ \hat{f}((0, 1), 2)z^2 \end{bmatrix} = \begin{bmatrix} (f(0, 0) + f(1, 0) + f(2, 0))z^0 \\ (f(0, 1) + f(1, 1) + f(2, 1))z^1 \\ (f(0, 2) + f(1, 2) + f(2, 2))z^2 \end{bmatrix}, \\
\hat{f}_z(1, 1) &= \begin{bmatrix} \hat{f}((1, 1), 0)z^0 \\ \hat{f}((1, 1), 1)z^1 \\ \hat{f}((1, 1), 2)z^2 \end{bmatrix} = \begin{bmatrix} (f(0, 0) + f(1, 2) + f(2, 1))z^0 \\ (f(0, 1) + f(1, 0) + f(2, 2))z^1 \\ (f(0, 2) + f(1, 1) + f(2, 0))z^2 \end{bmatrix}, \\
\hat{f}_z(2, 1) &= \begin{bmatrix} \hat{f}((2, 1), 0)z^0 \\ \hat{f}((2, 1), 1)z^1 \\ \hat{f}((2, 1), 2)z^2 \end{bmatrix} = \begin{bmatrix} (f(0, 0) + f(1, 1) + f(2, 2))z^0 \\ (f(0, 2) + f(1, 0) + f(2, 1))z^1 \\ (f(0, 1) + f(1, 2) + f(2, 0))z^2 \end{bmatrix}, \\
\hat{f}_z(1, 0) &= \begin{bmatrix} \hat{f}((1, 0), 0)z^0 \\ \hat{f}((1, 0), 1)z^1 \\ \hat{f}((1, 0), 2)z^2 \end{bmatrix} = \begin{bmatrix} (f(0, 0) + f(0, 1) + f(0, 2))z^0 \\ (f(1, 0) + f(1, 1) + f(1, 2))z^1 \\ (f(2, 0) + f(2, 1) + f(2, 2))z^2 \end{bmatrix}.
\end{align*}
\]
which gives the following matrix representation of the discrete Radon transform:

$$
\begin{bmatrix}
\hat{f}(1, 0, 0) \\
\hat{f}(1, 0, 1) \\
\hat{f}(1, 0, 2) \\
\hat{f}(0, 1, 0) \\
\hat{f}(0, 1, 1) \\
\hat{f}(0, 1, 2) \\
\hat{f}(1, 1, 0) \\
\hat{f}(1, 1, 1) \\
\hat{f}(1, 1, 2) \\
\hat{f}(2, 1, 0) \\
\hat{f}(2, 1, 1) \\
\hat{f}(2, 1, 2)
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f(0, 0) \\
f(0, 1) \\
f(0, 2) \\
f(1, 0) \\
f(1, 1) \\
f(1, 2) \\
f(2, 0) \\
f(2, 1) \\
f(2, 2) \\
\end{bmatrix}.
\tag{16}
$$

If

$$
K_3 = \begin{bmatrix}
1 & 1 & 1 \\
1 & K_{22} & K_{23} \\
1 & K_{32} & K_{33}
\end{bmatrix}, \quad \text{and} \quad \tilde{K}_3 = \begin{bmatrix}
1 & K_{22} & K_{23} \\
1 & K_{32} & K_{33}
\end{bmatrix},
$$

then we obtain the following expression for 2D $K$-transform

$$
\begin{bmatrix}
F(0, 0) \\
F(1, 0) \\
F(2, 0) \\
F(0, 1) \\
F(0, 2) \\
F(1, 1) \\
F(2, 2) \\
F(2, 1) \\
F(1, 2)
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & K_{22} & K_{23} \\
1 & K_{32} & K_{33} \\
1 & K_{22} & K_{23} \\
1 & K_{32} & K_{33} \\
1 & K_{22} & K_{23} \\
1 & K_{32} & K_{33} \\
\end{bmatrix}
\begin{bmatrix}
\hat{f}(1, 0, 0) \\
\hat{f}(1, 0, 1) \\
\hat{f}(1, 0, 2) \\
\hat{f}(0, 1, 0) \\
\hat{f}(0, 1, 1) \\
\hat{f}(0, 1, 2) \\
\hat{f}(1, 1, 0) \\
\hat{f}(1, 1, 1) \\
\hat{f}(1, 1, 2) \\
\hat{f}(2, 1, 0) \\
\hat{f}(2, 1, 1) \\
\hat{f}(2, 1, 2)
\end{bmatrix}.
$$

As a result, fast calculation of 2D DRT requires the following steps:

1. **Interpretation** of the 2D signal $f(i_1, i_2)$ as the 1D polynomial–valued signal (see (12)).
2. **Calculation** of one NPT which is necessary to calculate the first subDRT

$$
\hat{f}_z(k_1) = \hat{f}_z(\alpha_1) = N_q \{ f_z(i_1) \} = \sum_{i_1=0}^{q-1} f_z(i_1) z^{k_1 i_1} = \sum_{p=0}^{q-1} \hat{f}(\alpha_1, p) z^p
$$

and calculation of the second (trivial) DRT $\hat{f}((1, 0), p) := \sum_{\pi(i_1=p)} f(i_1, i_2) = \sum_{i_2=0}^{q-1} f(p, i_2)$.

To calculate 2D DKT it is required one full 1D DKT

- $F(a \alpha_2) = K^{\alpha_2} \{ \hat{f}(\alpha_2, p) \} = \sum_{p=0}^{q-1} \hat{f}(\alpha_2, p) K(ap), \quad a = 0, 1, \ldots, q - 1$
- and $q$ 0–reduced 1D DKTs (see (10))
Note that all the rays in (18) the ray $\text{St}_n$ where $K$ where

In this case the spectrum $F(k_1, \ldots, k_n)$ has the following form:

where $F$ and $K$ we can limit the spectrum $F(a \alpha_1)$, where $a \neq 0$. In this case 2D DKT is reduced to one NFT and $q$ 1D DKTs.

V. FAST nD RADON AND $\mathcal{K}$–TRANSFORMS

A. Fast nD $\mathcal{K}$–transform on $\mathbb{Z}_q^n$

The purpose of this subsection is to study the fast multidimensional DRT and DKT for $N = q$ in more details. For nD DKT on $\mathbb{Z}_q^n$ we have

$$F(k_1, \ldots, k_n) = \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} \cdots \sum_{i_n=0}^{q-1} f(i_1, \ldots, i_n) K(k_1i_1 + \cdots + k_ni_n).$$

Theorem 11: If $N = q$ is a prime, then $|\text{St}| = (q^n - 1)/(q - 1)$, and the star consists of $n$ substars $\text{St} = \bigcup_{r=1}^{n} \text{St}_r$, where

$$\text{St}_r = \{\alpha_r\} = \{(k_1, \ldots, k_{r-1}, 1, 0, \ldots, 0)\}, \quad k_r \in \mathbb{Z}_q.$$  

In this case the nD $\mathcal{K}$–transform $\mathcal{K}_{(q)}^n$ can be computed by the $(q^n - 1)/(q - 1)$ 1D $\mathcal{K}$–transforms $\mathcal{K}_q$ and by the one discrete nD Radon transform:

$$\mathcal{K}_{(q)}^n \{f(i_1, \ldots, i_n)\} = \left[ \mathcal{K}_q^n \oplus \bigoplus_{r=1}^{n-1} \mathcal{K}_q^n \right] \mathcal{R}_{(q)}^n \{f(i_1, \ldots, i_n)\}, \quad (17)$$

where $\mathcal{K}_q$ and $\tilde{\mathcal{K}}_q$ are full and $p$–reduced 1D $\mathcal{K}_q$–transforms, respectively.

Proof: Indeed, we can present the spectral domain in the following way:

$$\mathbb{Z}_q^n = \mathbb{Z}_q^{n-1} \times (\mathbb{M}Z_q \cup \{0\}) = [\mathbb{Z}_q^{n-1} \times \mathbb{M}Z_q] \cup (\mathbb{Z}_q^{n-1} \times \{0\}) =$$

$$= [\mathbb{Z}_q^{n-1} \times \mathbb{M}Z_q] \cup [\mathbb{Z}_q^{n-2} \times \mathbb{M}Z_q \times \{0\}] \cup (\mathbb{Z}_q^{n-2} \times \{0\}^2) =$$

$$= [\mathbb{Z}_q^{n-1} \times \mathbb{M}Z_q] \cup [\mathbb{Z}_q^{n-2} \times \mathbb{M}Z_q \times \{0\}] \cup [\mathbb{Z}_q^{n-3} \times \mathbb{M}Z_q \times \{0\}^2] \cup (\mathbb{Z}_q^{n-3} \times \{0\}^3) =$$

$$= \bigcup_{r=1}^{n} [\mathbb{Z}_q^{n-r} \times \mathbb{M}Z_q \times \{0\}^{r-1}] = \bigcup_{r=1}^{n} \text{Ray}_{\mathbb{Z}_q}(k_1, \ldots, k_{r-1}, 1, 0, \ldots, 0) =$$

$$= \bigcup_{r=1}^{n} \text{Ray}_{\mathbb{Z}_q}(\text{St}_r) = \bigcup_{r=1}^{n} \text{Ray}_{\mathbb{M}Z_q}(\text{St}_r), \quad (18)$$

where $\text{St}_n := (1_n, 0, \ldots, 0)$, $\text{St}_r := \{\alpha_r = (k_1, \ldots, k_{r-1}, 1, 0, \ldots, 0) \in \mathbb{Z}_q^{n-r} \times \mathbb{M}Z_q \times \{0\}^{r-1}\}$. Note that all the rays $\text{Ray}_{\mathbb{Z}_q}(\alpha_r)$ over $\mathbb{Z}_q$ are intersected in the origin. Hence, we have to take in (18) the ray $\text{Ray}_{\mathbb{Z}_q}(\text{St}_n)$ over $\mathbb{Z}_q$ and take all the rays $\text{Ray}_{\mathbb{Z}_q}(\text{St}_r)$, $r = 1, \ldots, n - 1$, only over $\mathbb{M}Z_q$. Using (9) and (18), we obtain (17).

From (18) we see that nD DRT split in $n$ discrete Radon subtransforms (subDRT) associated with every substar. Every subDRT has the following form:

$$\tilde{f}(\alpha_r, p) = \sum_{k_1i_1 + \cdots + k_{r-1}i_{r-1} + \cdots + k_r} \sum_{i_r} \sum_{i_{r+1}} \cdots \sum_{i_n} f(i_1, \ldots, i_{r-1}, i_r, i_{r+1} \ldots, i_n) =$$
\[
\begin{align*}
= \sum_{k_1 i_1 + \ldots + k_r i_r - 1 + \ldots + i_r} \sum_{i_r=0}^{q-1} \left( \sum_{i_{r+1}=0}^{q-1} \sum_{i_n=0}^{q-1} f(i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_n) \right) \\
= \sum_{k_1 i_1 + \ldots + k_r i_r - 1 + \ldots + i_r} \sum_{i_r=0}^{q-1} f^{(r)}(i_1, \ldots, i_{r-1}, i_r), \quad r = 1, \ldots, n,
\end{align*}
\]

where

\[
f^{(r)}(i_1, \ldots, i_r) = \sum_{i_r=0}^{q-1} f(i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_n), \quad r = 1, \ldots, n.
\]

How can we efficiently calculate the DRT (19)? For fast calculation of sums (19) we will interpret the rD scalar–valued signal \( f^r(i_1, i_2, \ldots, i_r, i_r, i_{r+1}, \ldots, i_n) \) as \((n-1)\)-dimensional polynomial–valued signal:

\[
f_z(i_1, \ldots, i_{r-1}, i_r+1, \ldots, i_n) = \sum_{i_r=0}^{q-1} f(i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_n) z^{i_r}
\]

having components from the polynomial ring \( R[z]/(z^q - 1) \), i.e.,

\[
f_z(i_1, \ldots, i_n) : \mathbb{Z}^n_q \rightarrow R[z]/(z^q - 1).
\]

The space of these signals will be denoted by \( L(\mathbb{Z}^n_q, R[z]/(z^q - 1)) \). In this space we introduce the polynomial–valued basis

\[
E^{(k_1, \ldots, k_r-1, k_r+1, \ldots, k_n)}(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_n) = z^{k_1 i_1 + \ldots + k_{r-1} i_{r-1} + k_r i_{r+1} + \ldots + k_n i_n},
\]

where \( k_1, i_1, \ldots, k_n, i_n \in \mathbb{Z}_q \).

The polynomial–valued spectrum \( \hat{f}_z(k_1, \ldots, k_{r-1}, k_r+1, \ldots, k_n) \) of the polynomial–valued signal \( f_z(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_n) \) is

\[
\hat{f}_z(k_1, \ldots, k_{r-1}, k_r+1, \ldots, k_n) = \\
= \sum_{i_r=0}^{q-1} \sum_{i_{n-1}=0}^{q-1} f_z(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_n) z^{k_1 i_1 + \ldots + k_{r-1} i_{r-1} + k_r i_{r+1} + \ldots + k_n i_n}.
\]

The geometrical nature of the polynomial–valued spectrum \( \hat{f}_z(k_1, \ldots, k_{r-1}, k_r+1, \ldots, k_n) \) is obvious after substituting (21) into (22):

\[
\begin{align*}
\hat{f}_z(k_1, \ldots, k_{r-1}, k_r+1, \ldots, k_n) & = \\
= \sum_{i_1=0}^{q-1} \sum_{i_r=0}^{q-1} f(i_1, \ldots, i_r, i_{r+1}, \ldots, i_n) z^{k_1 i_1 + \ldots + k_{r-1} i_{r-1} + i_r + k_r i_{r+1} + \ldots + k_n i_n} \\
& = \sum_{p=0}^{q-1} \left( \sum_{k_1 i_1 + \ldots + k_{r-1} i_{r-1} + i_r + k_r i_{r+1} + \ldots + k_n i_n = p} \sum_{i_1=0}^{q-1} \sum_{i_r=0}^{q-1} \sum_{i_{n-1}=0}^{q-1} \sum_{i_{r+1}=0}^{q-1} \sum_{i_n=0}^{q-1} \right) z^p.
\end{align*}
\]

Therefore, the coefficients of \( \hat{f}_z(k_1, \ldots, k_{r-1}, k_r+1, \ldots, k_n) \) are the spectral components of the Radon transform

\[
\hat{f}(\alpha_r, p) = \sum_{(\alpha_r, i_r) = p} f(i_1, \ldots, i_r) = \sum_{k_1 i_1 + \ldots + k_{r-1} i_{r-1} + k_r = p} \sum_{i_1=0}^{q-1} \sum_{i_r=0}^{q-1} f(i_1, \ldots, i_r).
\]
of the signal $f^{(r)}(i_1, \ldots, i_r)$ on the hyperplanes $(\alpha_r | i) = p$, and these coefficients we can calculate using the fast algorithm of NPT (22). Indeed, it is easy to see, that this transform is separable. Hence, it can be described in the operator notation as

$$\hat{f}_z(\alpha_r, p) = \left( \bigotimes_{l=1}^{r-1} \mathcal{N}_q^{(l)} \right) \left( \bigotimes_{l=r+1}^{n} \mathcal{N}_q^{(l)} \right) \left\{ f_z(i_1, \ldots, i_r, i_{r+1}, \ldots, i_n) \right\} =$$

$$= \left( \bigotimes_{l=1}^{r-1} \mathcal{N}_q^{(l)} \right) \left( \bigotimes_{l=r+1}^{n} \mathcal{S}_q^{(l)} \right) \left\{ f_z(i_1, \ldots, i_r, i_{r+1}, \ldots, i_n) \right\} = \left( \bigotimes_{l=1}^{r-1} \mathcal{N}_q^{(l)} \right) \left\{ f_z^{(r)}(i_1, \ldots, i_{r-1}) \right\} =$$

$$= \prod_{l=1}^{r-1} \left( I_{p^{l-1}} \otimes \mathcal{N}_q^{(l)} \otimes I_{q^{r-l}} \right) \left\{ f_z^{(r)}(i_1, \ldots, i_{r-1}) \right\}, \quad (24)$$

where $\mathcal{N}_q^{(l)}$ and $\mathcal{S}_q^{(l)}$ are 1D $q$–point NPT and 1D $q$–point operator of the summation acting along the $l$th coordinate, respectively, and

$$f_z^{(r)}(i_1, \ldots, i_{r-1}) := \left( \bigotimes_{l=r+1}^{n} \mathcal{S}_q^{(l)} \right) \left\{ f_z(i_1, \ldots, i_r, i_{r+1}, \ldots, i_n) \right\} =$$

$$= \left( \bigotimes_{l=r+1}^{n} \mathcal{N}_q^{(l)} \right) \left|_{k_{r+1}=0, \ldots, k_n=0} \left\{ f_z(i_1, \ldots, i_r, i_{r+1}, \ldots, i_n) \right\}. \quad (24)$$

The Expression (24) means that $(r-1)$–dimensional Nüsbaumer transform has radix–$q$ fast transform algorithm. For this transform we need $(n-r)q^{n-r-1}$ of 1D fast NPT $\mathcal{N}_q$. The total number of the fast 1D NPTs equals $\sum_{r=1}^{n-1}(n-r)q^{n-r-1} = q^{n-1}[n(q-1)-q+1]/(q-1)^2$ and if $q > 2$ this number is approximately equals $nq^{n-2}$.

From the polynomial–valued spectrum we can obtain full $\mathcal{K}$–spectrum using one full 1D $\mathcal{K}$–transform

$$F(a\alpha_n) = K^{\alpha_n} \{ \hat{f}(\alpha_n, p) \} = \sum_{p=0}^{q-1} \hat{f}_r(a\alpha_n, p)K(ap), \quad a = 0, 1, \ldots, p-1$$

and using $|St| - 1 = q(q^{n-1} - 1)/(q - 1)$ reduced 1D $\mathcal{K}$–transforms (see (17))

$$F(a\alpha_r) = K^{\alpha_r} \{ \hat{f}(\alpha_r, p) \} = \sum_{p=1}^{q-1} \hat{f}_r(a\alpha_r, p)K(ap), \quad a = 1, 2, \ldots, p-1,$$

for all $r = 1, 2, \ldots, n-1$. Hence, if $a \neq 0$ we obtain $nD$ $\mathcal{K}$–spectrum lying on the rays $\text{Ray}(\alpha_r)$ from $\hat{f}_r(\alpha_r, p)$ using $q^{r-1}$ 1D DKTs (for every $r = 1, 2, \ldots, n-1$). We obtain full $nD$ $\mathcal{K}$–spectrum if $r$ runs from 1 to $n$ using $(q^n - 1)/(q - 1)$ 1D DKTs.

Recall (see [4]), that additive and multiplicative complexities of 1D NPT are $\text{Ad}(\mathcal{N}_q) = q \cdot \text{Ad}(\mathcal{F}_q)$, and $\text{Mu}(\mathcal{N}_q) = 0$, where $\text{Ad}(\mathcal{F}_q)$, $\text{Mu}(\mathcal{F}_q)$ are additive and multiplicative complexities of the fast 1D $q$–points DFT $\mathcal{F}_q$, respectively. The total complexity for computing $nD$ DRT is

$$\text{Ad}(\mathcal{R}_{(q)^n}) = \text{Ad} \left( \bigotimes_{l=r+1}^{n} \mathcal{S}_q^{(l)} \right) + \text{Ad} \left( \bigotimes_{l=1}^{r-1} \mathcal{N}_q^{(l)} \right) = q^{n-1} - 1 + nq^{n-2} \text{Ad}(\mathcal{N}_q),$$

$$\text{Mu}(\mathcal{R}_{(q)^n}) = 0,$$
and for computing $n$D DKT is

\[
\text{Ad}(\mathcal{K}(q)^n) = \frac{q^n - 1}{q - 1} + nq^{n-2}\text{Ad}(\mathcal{N}_q) + \frac{q^{n-1} - 1}{q - 1}\text{Ad}(\mathcal{K}_q),
\]

\[
\text{Mu}(\mathcal{K}(q)^n) = \frac{q^{n-1} - 1}{q - 1}\text{Mu}(\mathcal{K}_q).
\]

For example, for $n$D FFT we have

\[
\text{Ad}(\mathcal{F}(q)^n) \approx (n + 1)q^{n-1}\text{Ad}(\mathcal{F}_q), \quad \text{Mu}(\mathcal{F}(q)^n) \approx q^{n-1}\text{Mu}(\mathcal{F}_q).
\]

Therefore, additive complexity of the present algorithm and traditional algorithms are equivalent, but multiplicative complexity of the new algorithm is $n$ times smaller comparing to the classical row/column Cooley’s–Tukey’s approach.

**B. Fast $n$D $K$–transform on $\mathbb{Z}_{q^m}^n$**

For $n$D DKT on $\mathbb{Z}_{q^m}^n$ we have

\[
F(k_1, \ldots, k_n) = \sum_{i_1=0}^{q^m-1} \sum_{i_2=0}^{q^m-1} \cdots \sum_{i_n=0}^{q^m-1} f(i_1, \ldots, i_n)K(k_1i_1 + \ldots + k_ni_n).
\]

**Theorem 12:** If $N = q^m$, where $q$ is a prime, then

\[
|\text{St}| = \frac{(q^{m(n-1)} - 1)}{(q^{n-1} - 1)} \cdot \frac{(q^n - 1)}{(q - 1)}
\]

and the star $\text{St}$ consists of $mn$ substars:

\[
\text{St} = \bigcup_{r=1}^{n} \bigcup_{s=1}^{m} \text{St}_r^s = \bigcup_{r=1}^{n} \bigcup_{s=1}^{m} \{\alpha_r^s\},
\]

where

\[
\text{St}_r^m = \{\alpha_r^m\} := \{(k_1^m, \ldots, k_{r-1}^m, 1_r, \ldots, 1_{n-r+1}, d^m_{r+1}, \ldots, d^m_n)\},
\]

\[
\text{St}_r^2 = \{\alpha_r^2\} := \{(k_1^2, \ldots, k_{r-1}^2, 1_r, \ldots, 1_{n-r+1}, d^2_{r+1}, \ldots, d^2_n)\},
\]

\[
\text{St}_r^1 = \{\alpha_r^1\} := \{(k_1^1, \ldots, k_{r-1}^1, 1_r, \ldots, 1_{n-r+1}, d^1_{r+1}, \ldots, d^1_n)\},
\]

$k_r^s \in \mathbb{Z}_{q^s}$, $d_r^s \in q\mathbb{Z}_{q^s}$, $s = 1, \ldots, m$ and $q\mathbb{Z}_{q^s}$ is the set of divisors of zero belonging to the ring $\mathbb{Z}_{q^s}$. In this case $n$D $K$–transform $\mathcal{K}(q^m)^n$ can be computed by the $|\text{St}|$ 1D $K$–transforms $\mathcal{K}_q^m$ and by the one discrete $n$D Radon transform, i.e.,

\[
\mathcal{K}(q^m)^n \{f(i_1, \ldots, i_n)\} = \left[\mathcal{K}_q^m \oplus \bigoplus_{r=1}^{n-m} \bigoplus_{s=1}^{m} \mathcal{K}_q^{s\oplus} \bigoplus_{r=1}^{n-m} \mathcal{K}_q^{s\ominus} \bigoplus_{r=1}^{n-m} \mathcal{R}_q^{(q^m)^n} \{f(i_1, \ldots, i_n)\} \right],
\]

where $\mathcal{K}_q^m$ and $\mathcal{K}_q^{s\oplus}$ are full and $q$–reduced 1D $\mathcal{K}_q^{s\ominus}$–transform, respectively.

**Proof:** The ring $\mathbb{Z}_{q^m}$ consists of numbers of two types: numbers $a$ relatively prime with $q$ (i.e., $(a, q^m) = 1$) and divisors of zero $d$, having form $d = qb \in q\mathbb{Z}_{q^m}$, $b \in \mathbb{Z}_{q^m-1}$. The numbers $a$ form the multiplicative group $\mathbb{M}\mathbb{Z}_{q^m}$ with $q^m - q^{m-2}$ elements and the numbers $d$ form a set of zero divisors $q\mathbb{Z}_{q^m}$ with $q^{m-1}$ elements. Hence, the ring $\mathbb{Z}_{q^m}$ is the union of the multiplicative
group $MZ_q$ and the set of zero divisors $qZ_{qm}$, i.e., $Z_{qm} = MZ_{qm} \cup qZ_{qm}$. Thus, we can present the spectral domain as the following union:

$$Z_{qm}^n = Z_{qm}^{n-1} \times Z_{qm} = Z_{qm}^{n-1} \times (M_{Z_{qm}} \cup qZ_{qm}) = (Z_{qm}^{n-1} \times M_{Z_{qm}}) \cup (Z_{qm}^{n-1} \times qZ_{qm}) =$$

$$= [Z_{qm}^{n-1} \times M_{Z_{qm}}] \cup [Z_{qm}^{n-2} \times M_{Z_{qm}} \times qZ_{qm}] \cup (Z_{qm}^{n-2} \times qZ_{qm}^2) =$$

$$= [Z_{qm}^{n-1} \times M_{Z_{qm}} \times qZ_{qm}^0] \cup [Z_{qm}^{n-2} \times M_{Z_{qm}} \times qZ_{qm}^1] \cup [Z_{qm}^{n-3} \times M_{Z_{qm}} \times qZ_{qm}^2] \cup (Z_{qm}^{n-3} \times qZ_{qm}^3) = \ldots$$

$$= \bigcup_{r=1}^{n-1} [Z_{qm}^{n-r} \times M_{Z_{qm}} \times qZ_{qm}^{r-1}] = \bigcup_{r=1}^{n-1} [Z_{qm}^{n-r} \times M_{Z_{qm}} \times qZ_{qm}^{r-1}] \cup qZ_{qm}^r. \quad (29)$$

Note that the $nD$ cube $qZ_{qm}^n$ is, in fact, the decimated version by the factor $q$ in all coordinate directions of the initial cube, i.e., $qZ_{qm}^n = Z_{qm}^{n-1}$. This brings us back to the initial problem for the cube $Z_{qm}^n$. Continuing the decomposition process (29) iteratively for the cubes $Z_{qm}^{n-1}$, $Z_{qm}^{n-2}, \ldots, Z_{q2}, Z_{q1}$, we obtain

$$Z_{qm}^n = \bigcup_{s=m}^{2} \bigcup_{r=1}^{n-1} [Z_{qm}^{n-r} \times M_{Z_{qm}} \times qZ_{qm}^{r-1}] \cup Z_{qm}^s.$$ 

But, according to (18), $Z_{qm}^n = \bigcup_{r=1}^{n} [Z_{qm}^{n-r} \times M_{Z_{qm}} \times \{0\}^{r-1}]$. Hence,

$$Z_{qm}^n = \bigcup_{s=m}^{2} \bigcup_{r=1}^{n-1} [Z_{qm}^{n-r} \times M_{Z_{qm}} \times qZ_{qm}^{r-1}] \cup MZ_{qm}^0 \times 0^{n-1} = \text{Ray}_Z(\text{St}_n^1) \cup \bigcup_{s=1}^{m} \bigcup_{r=1}^{n-1} \text{Ray}_Z(\text{St}_r^s), \quad (30)$$

where $Z_{qm}^{r-1} := \{0\}^{r-1}$, $\text{St}_n^1 := \alpha_n^1 := (1_n, 0, 0, \ldots, 0) \in MZ_{qm}^n \times 0^{n-1}$,

$$\text{St}_r^s := \{\alpha_r^s := (k_1^s, \ldots, k_{r-1}^s, 1, r, d_{r+1}, \ldots, d_n) \in Z_{qm}^{n-r} \times M_{Z_{qm}} \times D_{Z_{qm}}^{r-1}, \text{and } r = 1, \ldots, n, s = 1, \ldots, m.$$ 

Now, combining (9) and (30), we obtain (28).

\[ \square \]

\text{From (30) we see that in this case DRT split in mn subDRT. Every subDRT has the following form:}

$$\hat{f}(\alpha_r^s, p) = \sum_{k_1^s, k_2^s} \ldots \sum_{k_{r-1}^r, k_{r+1}, \ldots, k_n} \sum_{i_r^s, i_{r+1}, \ldots, i_n} \sum_{i_1^s} f(i_1^s, \ldots, i_{r-1}^s, i_r^s, i_{r+1}, \ldots, i_n) \hat{z}^{i_r^s},$$

where $r = 1, \ldots, n$ and $s = 1, \ldots, m$. How can we efficiently calculate these subDRT? For fast calculation of these subDRT we will again interpret the nD scalar-valued signal $f(i_1^s, i_2^s, \ldots, i_n^s)$ as $(n-1)$-dimensional polynomial-valued signal:

$$f_z(i_1^s, \ldots, i_{r-1}^s, i_r^s, i_{r+1}, \ldots, i_n^s) = \sum_{i_r^s=0}^{q-1} f(i_1^s, \ldots, i_{r-1}^s, i_r^s, i_{r+1}, \ldots, i_n^s) z^{i_r^s}, \quad (31)$$

having components from the polynomial ring $R[z]/(z^{q^s} - 1)$:

$$f_z(i_1^s, \ldots, i_{r-1}^s, i_r^s, i_{r+1}, \ldots, i_n^s) : Z_{qm}^{n-1} \rightarrow R[z]/(z^{q^s} - 1).$$

The space of these signals will be denoted by $L(Z_{qm}^{n-1}, R[z]/(z^{q^s} - 1))$. In this space we introduce the polynomial-valued basis

$$E_2^{(k_1^s, \ldots, k_{r-1}^s, k_{r+1}^s, \ldots, k_n^s)}(i_1^s, \ldots, i_{r-1}^s) = z^{i_1^s + \ldots + k_{r-1}^s i_{r-1}^s + k_{r+1}^s i_{r+1}^s + \ldots + k_n^s i_n^s},$$

where $k_1^s, i_1^s, \ldots, k_n^s, i_n^s \in Z_{qm}^s$. 


The polynomial–valued spectrum \( \hat{f}_z(k_1^s, \ldots, k_{r-1}^s, k_{r+1}^s, \ldots, k_n^s) \) of the polynomial–valued signal \( f_z(i_1^s, \ldots, i_r^s, \ldots, i_r^s, \ldots, i_n^s) \) is

\[
\hat{f}_z(k_1^s, \ldots, k_{n-1}^s, k_n^s) = \\
= \sum_{i_1^s=0}^{q^s-1} \ldots \sum_{i_{r-1}^s=0}^{q^s-1} f_z(i_1^s, \ldots, i_{r-1}^s, i_{r+1}^s, \ldots, i_n^s) z^{k_1^s i_1^s + \cdots + k_{r-1}^s i_{r-1}^s + i_{r+1}^s i_{r+1}^s + \cdots + k_n^s i_n^s},
\]

for every 1D \( -r \)–...–...–\( -r \)–...–...–1–\( n \)–...–...–1–...1–\( n \)–...–...–1. Every 1D NPT \( \mathcal{N}_q^s \) is full and 0-reduced 1D NPT acting along the \( l \)th coordinate. Every 1D NPT \( \mathcal{N}_q^s \) is the polynomial \((q^s \times q^s)\)–transform in the basis of the polynomial–valued functions \( z^{ik} \mod 2^{q^s} = 0 \). This fact means, that this transform has the Fourier–like form (instead of \( e^{j z^{ik} k} \) we have \( z^{ik} \)) and, hence, this transform has the classical radix–q Cooley–Tukey factorization [1],[44]–[45].

\[
\mathcal{N}_q^s = \Pi_{q^s}^{(2)} \prod_{i=1}^{s} \left( I_{q^s} \odot \Pi_{q^{s-1}}^{(2)} \right) \prod_{k=1}^{s-i} \left( I_{q^s} \odot D_q(z^{q^s-i-k-1}) \odot I_{q^s-i-k-1} \right),
\]

where \( \Pi_{q^s}^{(2)} \) is the \( q \)–ary reversal permutation matrix, and

\[
D_q(z^{q^s-i-k-1}) := \text{diag}(1, z^1 q^{n-i-k-1}, z^2 q^{q^s-i-k-1}, \ldots, z^{q-1} q^{q^s-i-k-1})
\]
are diagonal matrices of twiddle factors. The Expressions (34) and (35) mean that sD Nusbaumer transform has the fast $q$–radix transform. Their basic operation is polynomial “Cooley–Tukey butterfly operation” with polynomial twiddle factors of the type $z^k$. This factor cycle permute coefficients of the polynomial–valued data of NPT and, hence, this fast algorithm is performed without multiplications. The most interesting case is when $q = 2$. In this case algorithm possesses a very regular radix–2 structure.

From the polynomial–valued spectrum we can obtain full $K$–spectrum using full 1D $K$–transform

$$ F(a\alpha^k_n) = K\alpha^k_n \{ f_r(\alpha^l_n, p) \} = \sum_{p=0}^{q-1} f_r(\alpha^l_n, p) K(ap), \quad a \in \mathbb{Z}_q $$

and using $q$–reduced 1D $K$–transforms (see (28))

$$ F(a\alpha^k_s) = K\alpha^k_s \{ f_r(\alpha^s_s, p) \} = \sum_{p=0}^{q^s-1} f_r(\alpha^s_s, p) K(ap), \quad a \in q\mathbb{Z}_{q^s}, $$

for all $r = 1, \ldots, n, \ s = 1, \ldots, m$.

C. Fast nD $K$–transform on $\mathbb{Z}^m_{q_1^{m_1} q_2^{m_2} \cdots q_l^{m_l}}$

For nD DKT on $\mathbb{Z}^m_N$ we have

$$ F(k_1, \ldots, k_n) = \sum_{i_1=0}^{N-1} \cdots \sum_{i_n=0}^{N-1} f(i_1, \ldots, i_n) K(k_1i_1 + \cdots + k_ni_n), $$

where $N = q_1^{m_1} q_2^{m_2} \cdots q_l^{m_l}$.

**Theorem 13:** If $N = q_1^{m_1} q_2^{m_2} \cdots q_l^{m_l}$, where $q_1, q_2, \ldots, q_l$ is a coprimes, then

$$ |S_t| = \prod_{i=1}^{l} \frac{(q_i^{m_i(n-1)} - 1)}{(q_i^{m_i-1} - 1)} \cdot \frac{(q_t^n - 1)}{(q_t - 1)} $$

and the star $S_t$ is the direct sum of $l$ substars (26) $S_t = \bigoplus_{t=1}^{l} (t) S_t$. In this case nD $K$–transform $K_{N^n}$ split on $l$ parallel/independent nD $K$–transforms $K_{(q_t^{m_t})^n}$ ($t = 1, \ldots, l$) of the form (28):

$$ K_{N^n} \{ f(i_1, \ldots, i_n) \} = \bigoplus_{t=1}^{l} K_{(q_t^{m_t})^n} \{ f(i_t^1, \ldots, i_t^n) \}, $$

where $i_t^1 := i_1 \ (mod \ p_t^{m_t})$, \ldots, $i_t^n := i_n \ (mod \ p_t^{m_t})$, $t = 1, \ldots, l$.

**Proof:** In this case the signal domain is nD cube over $\mathbb{Z}^m_{q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}}$, i.e. $Sp(\mathbb{Z}^m_{q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}}) = \mathbb{Z}^m_{q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}}$. According to the Chinese remainder theorem [1], we have $\mathbb{Z}^m_{q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}} = \bigoplus_{t=1}^{l} \mathbb{Z}^m_{q_t^{m_t}}$. Hence,

$$ Sp(\mathbb{Z}^m_{q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}}) = \left[ \bigoplus_{t=1}^{l} \mathbb{Z}^m_{q_t^{m_t}} \right]^n = \bigoplus_{t=1}^{l} \mathbb{Z}^m_{q_t^{m_t}}. $$

Consequently,

$$ S_t = \bigoplus_{t=1}^{l} (t) S_t, \quad |S_t| = \prod_{i=1}^{l} \frac{(q_t^{m_t(n-1)} - 1)}{(q_t^{m_t-1} - 1)} \cdot \frac{(q_t^n - 1)}{(q_t - 1)} $$

and nD $K$–transform $K_{N^n}$ split on $l$ parallel/independent nD $K$–transforms $K_{(q_t^{m_t})^n}$. □
Note, that if \( nD \mathcal{K} \)-transform \( \mathcal{K}_{N^n} \) is separable on \( \mathbb{Z}_N^n \) then the classical “row/column” algorithm reduces this transform to \( nN^{n-1} \) 1D \( \mathcal{K} \)-transforms \( \mathcal{K}_N \) of the length \( N \) [1]–[3]. From the Theorems 2–4 it follows that \( nD \mathcal{K} \)-transform \( \mathcal{K}_{N^n} \) is a composition of \( nD \) DRT \( R_{N^n} \) and a set of \( |St| \approx N^{n-1} \) parallel/independ 1D \( \mathcal{K} \)-transforms \( \mathcal{K}_N \), leading to decrease of a total number of 1D \( \mathcal{K} \)-transforms by the factor of \( n \) comparing to the classical “row/column” approach.

VI. Conclusions

The important contribution of this work is that it brings a new approach to independent/parallel decomposition of a class of \( nD \) non–separable discrete unitary \( \mathcal{K} \)-transforms. These transforms possess an interesting structure which is utilized to reduce their computational complexity. This approach has the following properties. It

- gives independent/parallel decomposition for a wide class of non–separable \( nD \) DKT,
- requires fewer 1D DOKTs than the classical separable radix FFT–type approach,
- has \( n \) times smaller multiplicative complexity of the classical separable DKT (the discrete Fourier, Hartley and Walsh transforms, the discrete cosine and sine transforms, etc.) comparing to the ”row/column” approach,
- introduces new fast direct and exact inversion algorithms of the discrete Radon transform.

REFERENCES

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